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7ème Cours

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filtrations: a general formalism

See Yves André "Slope filtrations" for a more general formalism.

$\mathcal{C}$ = exact category

$\mathcal{A}$ = abelian category

$F : \mathcal{C} \to \mathcal{A}$ exact faithful

"generic fiber functor"

s.t. $\forall X \in \mathcal{C}$

"schematical closure"

$F : \{\text{strict sub-objects of } X\} \xrightarrow{\sim} \{\text{sub-objects of } F(X)\}$

↳ the one that can be inserted in an exact sequence

* $rg : Ob(A) \to \mathbb{N}$ additive, $rg(X) = 0 \Leftrightarrow X = 0$

$\downarrow$

on exact sequences i.e. factorizes through $Ob(A) \to K_0(A) \to \mathbb{Z}$

$X \mapsto [X]$

group morphism

We still denote $rg : Ob(\mathcal{C}) \to \mathbb{N}$ for $rg \circ F$.

* $\deg : Ob(\mathcal{C}) \to \mathbb{R}$ additive

s.t. $\left[X \xrightarrow{u} Y \text{ iso. in generic fiber} \right.$

$\Downarrow$

$\left. \deg(X) \leq \deg(Y) \text{ with equality iff } u \text{ is an iso.} \right]$

Examples:

* X integral Dedekind scheme equipped with a degree fct. $\deg : |X| \to \mathbb{N}_{\geq 1}$ satisfying

$$\left[\ \forall f \in k(X)^\times \quad \deg(\operatorname{div} f) = 0\ \right]$$

$\left(k(X) = \mathcal{O}_{X,\eta} \text{ field of rational functions} \right)$

$\mathcal{C} = \mathrm{Fib}_X$ = vector bundles$/X$

$\mathcal{A} = \mathrm{Vect}_{k(X)}$ (finite dim. v.s.)

$$\begin{cases} F : \mathcal{C} \to \mathcal{A} \\ \quad \mathcal{E} \mapsto \mathcal{E}_\eta \end{cases}$$

strict sub-objects of $\mathcal{E}$
$\parallel$
locally direct factor $\mathcal{F} \subset \mathcal{E}$

$\{\text{strict sub ob. of } \mathcal{E}\} \xrightarrow{\sim} \{\text{sub. v.s. of } \mathcal{E}_\eta\}$

$\mathcal{E} \cap V \leftarrow\!\shortmid V$

inside $\mathcal{E}_\eta$

rk = usual rk. fct.

$$\mathrm{Pic}(X) = \mathrm{Div}(X)/\sim \ \xrightarrow{\deg} \mathbb{Z}$$

Weil-Divisors

rational equivalence

Induces $\deg : \mathrm{Bun}_X \to \mathbb{Z}$

$$\mathcal{E} \mapsto \deg(\det \mathcal{E})$$

Then $u : \mathcal{E} \to \mathcal{E}'$ induces $\mathcal{E}_\eta \xrightarrow{\sim} \mathcal{E}'_\eta$

$\Rightarrow \quad 0 \to \mathcal{E} \to \mathcal{E}' \to \underbrace{\mathcal{F}}_{\text{torsion coherent sheaf}} \to 0$

$$\deg(\mathcal{E}') = \deg(\mathcal{E}) + \underbrace{\deg(\mathcal{F})}_{\geq 0 \text{. This is } 0 \iff \mathcal{F} = 0.}$$

$$\mathcal{F} \simeq \bigoplus_{x \in |X|} i_{x*} \underbrace{M}_{\text{finite lenght } \mathcal{O}_{X,x}\text{-module}}$$

$$\deg(\mathcal{F}) = \sum_{x \in |X|} \text{lenght}(M) \cdot \deg(x)$$

* $\mathcal{C} = \mathcal{A}$ abelian category. $F = \mathrm{Id}$ functor

$\deg : \mathcal{A} \to \mathbb{R}$ additive + $\mathrm{rk}(X) = 0 \Leftrightarrow X = 0$.

$\mathrm{rk} : \mathcal{A} \to \mathbb{N}$

Ex: k field $BT_k \otimes \mathbb{Q}$ = abelian Cat. of p-div. gp./k up to isogeny.

$\mathrm{rk} := ht$ height of a p-div. gp.

$\deg := \underbrace{\dim}$

dimension of the associated formal group

$\mu = \frac{\deg}{\mathrm{rk}} \rightsquigarrow$ H.N. filtration = slope filtration

For example: $H \in BT_k$ $\quad 0 \to H^{\circ} \to H \to H^{\acute{e}t} \to 0$

is a part of this filtration.

* $L|K$ extension of fields

$$\mathcal{C} = \mathrm{VectFil}_{L|K} = \left\{ (V, \mathrm{Fil}^{\bullet} V_L) \;\middle|\; \begin{array}{l} V \in \mathrm{Vect}_K \\ \mathrm{Fil}^{\bullet} V_L \text{ decreasing} \\ \text{finite filtrations of } V \otimes_K L \end{array} \right\}$$

- Exact sequences: strictly compatible wrt. Filtration.

$\mathcal{A} = \mathrm{Vect}_K$

$$F : \mathcal{C} \to \mathcal{A}$$
$$(V, \mathrm{Fil}^\bullet V_L) \mapsto V$$

$$\begin{cases} \mathrm{rk} = \dim_K V \\ \deg = \sum_{i \in \mathbb{Z}} i . \dim_L \mathrm{gr}^i V_L \end{cases}$$ terminal point of Hodge polygon.

$$= \sum_{i > N} \dim \mathrm{Fil}^i V_L + N \dim V \quad \text{for } N \ll 0$$

$\hookrightarrow$ the property for degree function.

* $K_0 = W(k)_{\mathbb{Q}}$ k perfect
$K | K_0$

Fontaine's filtered φ-modules

$$\varphi\text{-ModFil}_{K/K_0} = \left\{ (D, \varphi, \mathrm{Fil}^\bullet D_K) \;\middle|\; \begin{array}{l} (D, \varphi) \text{ Isocrystal} \\ \mathrm{Fil}^\bullet D_K \text{ filtration of } D \otimes_{K_0} K \end{array} \right\}$$

$$\begin{cases} \mathrm{rk} := \dim_{K_0} D \\ \deg := t_H - t_N \end{cases}$$

semi-stable slope 0 objects
$\shortparallel$
weakly admissible
filtered φ-modules à la Fontaine

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* R = Bezout ring

$\mathcal{E} \subset R$

field valued, $v : \mathcal{E} \to \mathbb{R} \cup \{+\infty\}$

$\sigma \in \mathrm{End}(R)$ stabilizing $\mathcal{E}$ and such that $\forall x \in \mathcal{E}, v(\sigma(x)) = v(x)$

Hypothesis: * $\mathcal{E}^\times = R^\times$

* $\forall x \in R \setminus \{0\}$ s.t. $x^{\sigma-1} \in \mathcal{E}^\times$

one has $v\left(x^{\sigma-1}\right) \geq 0$

$$\mathcal{C} = \left\{ (M, \varphi) \;\middle|\; \begin{array}{l} M = \text{free } R\text{-module of finite type} \\ \varphi : M \to M \ \sigma\text{-linear} \\ \text{s.t. } \varphi \otimes \mathrm{Id} : M^{(\sigma)} \xrightarrow{\sim} M \end{array} \right\}$$

$$F(M, \varphi) = \left(M \underset{R}{\otimes} \mathrm{Frac}(R), \varphi \otimes \sigma \right)$$

~~rg~~ $\mathrm{rk} := \mathrm{rk}_R M$

$\deg(M, \varphi) = -v(a)$ if $\det(M, \varphi) = R.e$

with $\varphi(e) = \underset{\substack{\cap \\ R^\times = \mathcal{E}^\times}}{a}\, e$

Harder–Narasimhan filtrations

$\mathcal{C}$, $F: \mathcal{C} \to \mathcal{A}$, deg, rk. $\mu = \frac{\deg}{\mathrm{rk}}$

Def. $X \in \mathcal{C}$ is semi-stable if $\forall\, X' \subset X$ strict sub-object with $X' \neq 0$ one has $\mu(X') \leq \mu(X)$

Rem: Any morphism in $\mathcal{C}$ has a kernel and cokernel: if $f: X \to Y$ then $\ker f$ = schematical closure of $\ker(F(f))$

Notation: $X \subset Y$ will always mean X is a strict subobject of Y.

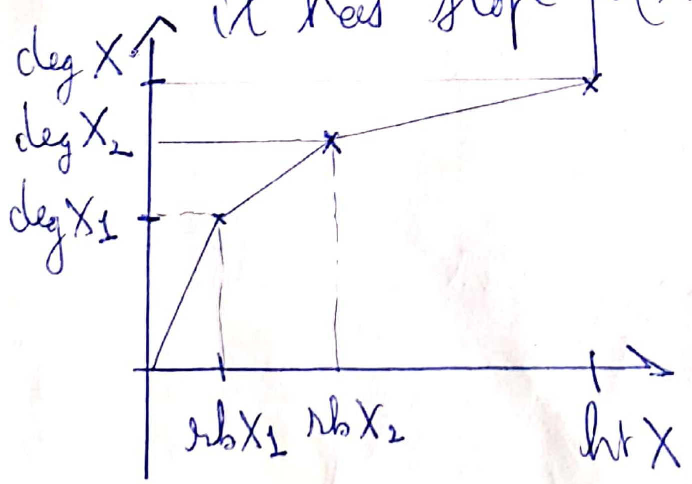
Th: $\forall X \in \mathcal{C}$ $\exists!$ filtration

$$0 = X_0 \subsetneq X_1 \subsetneq \dots \subsetneq X_n = X$$ such that:

* $\forall i \geq 1$, X_i / X_{i-1} is semi-stable
* $\mu(X_1/X_0) > \dots > \mu(X_n / X_{n-1})$

For such an X set

$HN(X)$ = Concave polygon defined on the segment $[0, \mathrm{rk} X]$ such that on the segment $[\mathrm{rk} X_i, \mathrm{rk} X_{i+1}]$ it has slope $\mu(X_i / X_{i+1})$



Th: $\forall Y \subset X$ the point $(\mathrm{rk}\, Y, \deg Y)$ is under the polygon $HN(X)$

$$\Rightarrow \left[HN(X) = \text{Concave envelope of the points } (\mathrm{rk}\, Y, \deg Y)_{Y \subset X}. \right]$$

Filtrations on $\mathcal{C}$: $\lambda \in \mathbb{R}$

$$\mathcal{C}^{\leq \lambda} = \{ X \in \mathcal{C} \,/\, \text{greatest slope of } X \text{ is } \leq \lambda \} = \{ X \,/\, \forall\, 0 \neq Y \subset X \;\; \mu(Y) \leq \lambda \}$$

$$\mathcal{C}^{\geq \lambda} = \{ X \in \mathcal{C} \,/\, \text{Smallest slope of } X \text{ is } \geq \lambda \}$$

$$= \{ X \,/\, \forall\, X \twoheadrightarrow Y \neq 0 \text{ epi. strict } \mu(Y) \geq \lambda \}$$

$(\mathcal{C}^{\leq \lambda})_{\lambda \in \mathbb{R}}$ $\downarrow$ filtration

$(\mathcal{C}^{\geq \lambda})_{\lambda \in \mathbb{R}}$ $\uparrow$ filtration

$$\mathcal{C}_\lambda^{ss} = \mathcal{C}^{\leq \lambda} \cap \mathcal{C}^{\geq \lambda}$$

$$= \{ X \text{ semi-stable slope } \lambda \}$$

Th: * $\mathcal{C}^{\leq\lambda}$ and $\mathcal{C}^{\geq\lambda}$ are sub-exact categories stable under extension in $\mathcal{C}$

* If $\lambda > \mu$ $\quad Hom(\mathcal{C}^{\geq\lambda}, \mathcal{C}^{\leq\mu}) = 0$

* $\mathcal{C}^{ss}_{\lambda}$ is an abelian category.

→ Devissage of the exact category $\mathcal{C}$ by the family of abelian categories $(\mathcal{C}^{ss}_{\lambda})_{\lambda\in\mathbb{R}}$

Proof of existence and unicity

* Preliminary remarks: $0 \to X' \to X \to X'' \to 0$

exact in $\mathcal{C}$ with $X' \neq 0$ and $X'' \neq 0$

$$\mu(X) = \frac{rk\, X'}{rk\, X}\mu(X') + \frac{rk\, X''}{rk\, X}\mu(X'')$$

barycenter

⇒ if $[a,b]$ = usual interval $[a,b]$ if $a \leq b$

$[b,a]$ if $a \geq b$

then $\mu(X) \in [\mu(X'), \mu(X'')]$.

Moreover if $\mu(X') \neq \mu(X'')$ then $\mu(X) \in]\mu(X'), \mu(X'')[$

Plus généralement, if $0 = X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n = X$

$$\mu(X) \in \mathrm{Conv}\left(\mu(X_i/X_{i-1})\right)_{1\leq i\leq n}$$

$$\Rightarrow \inf\{\mu(X_i/X_{i-1})\}_{1\leq i\leq n} \leq \mu(X) \leq \sup\{\mu(X_i/X_{i-1})\}_{1\leq i\leq n}$$

* Let $X \in \mathcal{C}$, $X \neq 0$. Consider the following property for $0 \neq Y \subsetneq X$

$$\left[\left[(SCSS) \quad Y \text{ is semi-stable and } \forall Y \subsetneq Y' \subset X \text{ one has } \mu(Y') < \mu(Y) \right]\right]$$

i.e. Y is a maximal ~~~~ semi-stable strict subobject.

Rem: The condition $Y \subsetneq Y' \subset X \Rightarrow \mu(Y') < \mu(Y)$

is equivalent to $0 \neq Y'' \subset X/Y \Rightarrow \mu(Y'') < \mu(Y)$.

(In fact if $Y'' = Y'/Y$

$\mu(Y') \in]\mu(Y), \mu(Y'')[$ since $\mu(Y) \neq \mu(Y'')$

$\Rightarrow \mu(Y') < \mu(Y)$.)

Lemma: $\exists$ at most one $Y \subset X$ satisfying (SCSS)

Proof: $Y_1, Y_2 \subset X$ satisfying (SCSS)

* To prove that $Y_1 = Y_2$ it suffices to prove that $Y_1 \subset Y_2$ or $Y_2 \subset Y_1$.

* Suppose $Y_1 \not\subset Y_2$. Consider the morphism

$$\begin{tikzcd}
Y_1 \arrow[rr, "f"] \arrow[dr] & & X/Y_2 \\
& \underbrace{\mathrm{Im} f}_{\text{Schematical closure of } \mathrm{Im}(F(f))} \arrow[ur, hook] &
\end{tikzcd}$$

$Y_1/\ker f \to \mathrm{Im} f$ iso in generic fiber

$\Rightarrow \mu(Y_1/\ker f) \leq \mu(\mathrm{Im} f)$

But Y_1 s.s $\Rightarrow \mu(\ker f) \leq \mu(Y_1)$ and thus since

$$\mu(Y_1) \in [\mu(\ker f), \mu(Y_1/\ker f)]$$

$$\mu(Y_1) \leq \mu(Y_1/\ker f)$$

(⚠ if $\ker f = 0$ but then is evident)

Then, Y_2 (scss) $\Rightarrow \mu(\mathrm{Im} f) < \mu(Y_2)$

Thus $\boxed{\mu(Y_1) < \mu(Y_2)}$

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By symmetry: if $Y_2 \not\subset Y_1$ then $\mu(Y_2) < \mu(Y_1)$

Thus, either $Y_1 \subset Y_2$ or $Y_2 \subset Y_1$ $\square$

Lemma: $\mu_{max}(X) = \sup\{\mu(Y) \mid Y \subset X, Y \neq 0\} < +\infty$

$\Rightarrow$ $rk : A \to \mathbb{N}$ satisfies $rk\, A = 0 \Leftrightarrow A = 0$

$\Rightarrow$ any object in $\mathcal{A}$ has finight lenght.

Let $0 = X_0 \subsetneq X_1 \subsetneq \dots \subsetneq X_n = X$ such that

$0 = F(X_0) \subsetneq \dots \subsetneq F(X_n) = F(X)$ is a Jordan-Hölder filtration

Let $0 \neq Y \subset X$ and $0 = Y_0 \subset Y_1 \subset \dots \subset Y_n = Y$ be the schematical closure of $(F(Y) \cap F(X_i))_{0 \leq i \leq n}$

$\forall i$ $\quad Y_i / Y_{i-1} \xrightarrow{u_i} X_i / X_{i-1}$ s.t.

$$F(u_i) : F(Y_i / Y_{i-1}) \hookrightarrow \underbrace{F(X_i / X_{i-1})}_{\text{simple object in } \mathcal{A}}$$

$\Rightarrow$ either $Y_i = Y_{i-1}$ either $F(u_i)$ is an iso.

$\Downarrow$

$\mu(Y_i / Y_{i-1}) \leq \mu(X_i / X_{i-1})$

Thus $\mu(Y) \leq \sup\{\mu(X_i/X_{i-1})\}_{1\leq i\leq n}$ $\square$

Lemma: The preceding bound is reached

(allways trivial if $\deg : C \to \mathbb{N}$ or $\{$sub-objects of $F(X)\}$ is finite)

$\to$ Exercise

Lemma: $\exists!\, Y \subset X$ (SCSS)

$\to$ Unicity allready seen. Then suffices to take Y of maximal rank among the $Y \subset X$ verifying $\mu(Y) = \mu_{max}(X)$ $\square$

Proof of the existence of the H.N. filtration:

$\leftrightarrow$ construct by induction $0 = X_0 \subsetneq \dots \subsetneq X_r = X$ s.t. $\forall i$ $X_{i+1}/X_i \subset X/X_i$ is (SCSS).

Proof of the unicity: It suffices to prove that if

$0 = X_0 \subsetneq \dots \subsetneq X_r = X$ is an H.N. filtration

then $X_1 \subset X$ is (SCSS)

~~Het~~

Let $Y \subset X/X_1$, $Y \neq 0$.

Let $0 = Y_1 \subset \dots \subset Y_n = Y$ be the filtration schematical closure of $\left(F(Y) \cap F(X_i/X_1)\right)_{1 \leq i \leq n}$

Let $u_i : Y_i/Y_{i-1} \to X_i/X_{i-1}$

$F(u_i)$ monomorphism. Thus $Y_i \neq Y_{i-1}$

$$\Rightarrow \mu(Y_i/Y_{i-1}) \leq \mu(\mathrm{Im}\, u_i) \underset{X_i/X_{i-1} \text{ s.s.}}{\leq} \mu(X_i/X_{i-1}) \underset{\text{since } \mu(X_1) > \mu(X_2/X_1) > \dots}{<} \mu(X_1)$$

Since $\mu(Y) \leq \sup\{\mu(Y_i/Y_{i-1}) \mid Y_i \neq Y_{i-1}\}_{1 \leq i \leq n}$

we thus have $\mu(Y) < \mu(X_1)$.

Thus, X_1 satisfies (SCSS) $\square$